MA 114 — Calculus II
Exam 3

Fall 2010
November 16, 2010

X130

This is a closed book exam. No books or notes are to be used during the exam. You may use a graphing calculator if it does not have symbolic manipulation capabilities. However, any) must be turned off and

vrite your solutions neatly
ustification will receive no
coherent written evidence
at are obtained simply as

$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\ln(\sec x + \tan x) = \sec x$$

Some error formulas:
If $|f''(x)| \le K$ for $a \le x \le b$, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \text{ and}$$
$$|E_M| \le \frac{K(b-a)^3}{24n^2}.$$

Problem	Score	Total
1	8	8
2	15	15
3	13	13
4	12	12
5	14	15
6	8	10
7	10	10
8	10	10
9	8	8
10	9	9
Total	108	110

1) (8 pts) Find the length of the curve $y = \frac{2}{3}(x-5)^{3/2}$, $5 \le x \le 8$.

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \frac{2}{3}(x-5)^{3/2} \qquad \frac{dy}{dx} = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)(x-5)^{1/2}(1) = (x-5)^{1/2}$$

$a = 5$
$b = 8$

$$A.L. = \int_5^8 \sqrt{1+\left((x-5)^{1/2}\right)^2}\,dx = \int_5^8 \sqrt{1+(x-5)}\,dx = \int_5^8 \sqrt{x-4}\,dx$$

Let $u = x-4$
$du = dx$

$$A.L. = \int_1^4 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{3}(x-4)^{3/2}\Big|_5^8$$

$$\int_a^b u^{1/2} = \frac{2}{3}u^{3/2}\Big|_a^b$$

$$A.L. = \left[\frac{2}{3}(8-4)^{3/2}\right] - \left[\frac{2}{3}(5-4)^{3/2}\right]$$

$$= \left[\frac{2}{3}(4)^{3/2}\right] - \left[\frac{2}{3}(1)^{3/2}\right] = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{16}{3} - \frac{2}{3}$$

$$= \boxed{\frac{14}{3} \text{ units}}$$

✓

2) (15 pts) Calculate the following integrals.

(a) (8 pts) $\int \frac{dx}{(1-x)\sqrt{x}} \Rightarrow$

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\cos^2(\theta) = 1 - \sin^2(\theta)$

let $\sqrt{x} = \sin(\theta)$ $\quad x = \sin^2(\theta)$ $\quad dx = 2\sin(\theta)\cos(\theta)$

then $1-x = 1-\sin^2(\theta) = \cos^2(\theta)$

$$\Rightarrow = \int \frac{2\sin(\theta)\cancel{\cos(\theta)}\,d\theta}{\cos^{\cancel{2}}(\theta)\,\cancel{\sin(\theta)}} = 2\int \frac{d\theta}{\cos(\theta)}$$

$$= 2\int \sec(\theta)\,d\theta = 2\ln|\sec(\theta)+\tan(\theta)| + C$$

1, $\sqrt{x}$, θ, $\sqrt{1-x}$

$\cos(\theta) = \sqrt{1-x}$

$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-x}}$

$\tan(\theta) = \frac{\sqrt{x}}{\sqrt{1-x}}$

$$= 2\ln\left|\frac{1}{\sqrt{1-x}} + \frac{\sqrt{x}}{\sqrt{1-x}}\right| + C = \boxed{2\ln\left|\frac{1+\sqrt{x}}{\sqrt{1-x}}\right| + C}$$

✓

~~Check:~~ $\cancel{\frac{\sqrt{1-x}}{1+\sqrt{x}} \cdot \frac{\sqrt{1-x}\left(\frac{1}{2\sqrt{x}}\right) - (1+\sqrt{x})\frac{1}{2\sqrt{1-x}}}{(1-x)} = \frac{\sqrt{1-x}}{1+\sqrt{x}} \cdot \frac{\frac{\sqrt{1-x}}{2\sqrt{x}} - \frac{1+\sqrt{x}}{2\sqrt{1-x}}}{\sqrt{(1-x)}} \left(\frac{\sqrt{1-x}}{1+\sqrt{x}} - \frac{1+\sqrt{x}}{\sqrt{1-x}}\right)\frac{1}{(1+\sqrt{x})(\sqrt{1-x})} = \frac{1}{\sqrt{x}(1+\sqrt{x})} - \frac{1}{(1-x)}}$

(b) (7 pts) $\int \frac{dx}{x(\ln x)^2}$

Let $u = \ln(x)$

$du = \frac{dx}{x}$

$$\int \frac{du}{u^2} = \int u^{-2}\,du$$

$$= -\frac{1}{u} + C$$

$$\int \frac{dx}{x(\ln(x))^2} = -\frac{1}{\ln|x|} + C$$

✓

Check: $-1[\ln(x)]^{-1}$

$1[\ln(x)]^{-2} \cdot \frac{1}{x}\,dx$

3) (13 pts) This problem concerns the improper integral $\int_0^1 \frac{dx}{(1-x)\sqrt{x}}$.

(Compare with problem 2a.)

(a) (10 pts) In order to determine whether this integral converges, two limits must be examined. Specify these two limits and compute each limit or show that it does not exist.

$$\lim_{a\to 0^+} \int_a^{1/2} \frac{dx}{(1-x)\sqrt{x}} + \lim_{b\to 1^-} \int_{1/2}^{b} \frac{dx}{(1-x)\sqrt{x}} \checkmark \qquad \int_a^b \frac{dx}{(1-x)\sqrt{x}} = 2\ln\left|\frac{1+\sqrt{x}}{\sqrt{1-x}}\right|\Big|_a^b$$

$$\lim_{a\to 0^+} 2\ln\left|\frac{1+\sqrt{x}}{\sqrt{1-x}}\right|\Big|_a^{1/2} = \lim_{a\to 0^+} 2\ln\left|\frac{1+\sqrt{1/2}}{\sqrt{1-1/2}}\right| - 2\ln\left|\frac{1+\sqrt{a}}{\sqrt{1-a}}\right| = 2\ln\left|\frac{1+\sqrt{1/2}}{\sqrt{1/2}}\right| - 2\ln|1|$$

$$= 2\ln\left|\frac{1+\sqrt{1/2}}{\sqrt{1/2}}\right| = 1.7627$$

$$\lim_{b\to 1^-} 2\ln\left|\frac{1+\sqrt{x}}{\sqrt{1-x}}\right|\Big|_{1/2}^{b} = \lim_{b\to 1^-} 2\ln\left|\frac{1+\sqrt{b}}{\sqrt{1-b}}\right| - 2\ln\left|\frac{1+\sqrt{1/2}}{\sqrt{1-1/2}}\right| = 2\ln\left|\frac{2}{0}\right| - 2\ln\left|\frac{1+\sqrt{1/2}}{\sqrt{1/2}}\right|$$ $\nearrow \infty$

limit diverges to ∞, DNE

$1.7627 + \infty \to \infty$

+10

(b) (3 pts) Does the given improper integral converge? Give your reason.

+3 The improper integral $\boxed{\text{DIVERGES}}$ because one of the limits of the proper integrals diverges. i.e, $1.7627 + \infty = \infty$

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4) (12 pts) This problem concerns the infinite series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

(a) (6 pts) Show that the series converges. If you use the integral test, your solution should verify that your choice of $f(x)$ satisfies the hypotheses of this test.

If $a_n = \frac{1}{n(\ln(n))^2}$ then let $f(x) = \frac{1}{x(\ln(x))^2}$ Then $f(x)$ resembles a_n

$f(x) \geq 0 \ \forall \ x \geq 2$ $\quad f(x) = (x)^{-1}(\ln(x))^{-2}$ $\quad f'(x) = \frac{1}{x} \cdot -2(\ln(x))^{-3} \cdot \frac{1}{x} + (\ln(x))^{-2}(-1)\frac{1}{x^2} = \frac{-2}{x^2 \ln(x)^3} - \frac{1}{x^2(\ln(x))^2}$

$f'(x) \leq 0 \ \forall \ x \geq 2 \therefore f(x)$ is decreasing

6

$$\lim_{b \to \infty} \int_2^b \frac{dx}{x(\ln(x))^2} = \lim_{b \to \infty} \int_{\ln(2)}^{\ln(b)} \frac{du}{u^2} = \lim_{b \to \infty} \frac{-1}{u}\Big|_{\ln(2)}^{\ln(b)} = \lim_{b \to \infty} \frac{-1}{\ln(x)}\Big|_2^b = \lim_{b \to \infty} \left[\frac{-1}{\ln(b)} + \frac{1}{\ln(2)}\right] = \boxed{\frac{1}{\ln(2)}}$$

$u = \ln(x)$ $\quad du = \frac{dx}{x}$

The integral converges to $\frac{1}{\ln(2)}$. Because $\lim_{b \to \infty} \int_2^b \frac{dx}{x(\ln(x))^2} \geq \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$, the series must also converge by the integral test.

(b) (6 pts) One part of the remainder estimate for the integral test states that under the hypotheses of the integral test, we have

$$0 \leq R_n \leq \int_n^{\infty} f(x)\,dx, \quad \text{where } R_n = s - s_n.$$

Use this inequality to determine the least number of terms of the series that need to be added to be sure that the sum is within 0.1 of the sum of the series.

$0.1 \geq \lim_{b \to \infty} \int_n^b \frac{dx}{x(\ln(x))^2}$ $\qquad \lim_{b \to \infty} \int_n^b \frac{dx}{x(\ln(x))^2} = \lim_{b \to \infty} \frac{-1}{\ln(x)}\Big|_n^b = \lim_{b \to \infty} \left[\frac{-1}{\ln(b)} + \frac{1}{\ln(n)}\right] = \frac{1}{\ln(n)}$

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$0.1 \geq \frac{1}{\ln(n)}$ $\qquad \ln(n) \geq \frac{1}{.1}$ $\qquad e^{\ln(n)} \geq e^{10}$ $\qquad n \geq 22026.46579$

$\boxed{n = 22027}$

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5) (15 pts) In this problem $f(x)$ is a function with values given in the following table.

x	-1.00	-.75	-.50	-.25	0.00	.25	.50	.75	1.0	1.25	1.50	1.75	2.00
$f(x)$	-16.	-9.	-6.0	-2.6	1.	4.6	8.0	11.	14.	14.	11.	-1.4	-26.

(a) (5 pts) Calculate the midpoint rule estimate M_3 for $\int_{-1}^{2} f(x)\,dx$.

$n=3 \quad \Delta x = \frac{2-(-1)}{3} = 1 \quad x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 2 \quad M_3 = \sum_{i=0}^{3} \Delta x\, f\left(\frac{x_i + x_{i+1}}{2}\right)$

$M_3 = 1\left(f\left(\frac{-1+0}{2}\right)\right) + 1\left(f\left(\frac{0+1}{2}\right)\right) + 1\left(f\left(\frac{1+2}{2}\right)\right) = f(-.5) + f(.5) + f(1.5) = -6 + 8 + 11 = \boxed{13 \text{ un}^2}$

(b) (5 pts) Calculate the trapezoidal rule estimate T_4 for $\int_{-1}^{2} f(x)\,dx$.

$n=4 \quad \Delta x = \frac{2-(-1)}{4} = 0.75 \quad x_1 = -1,\ x_2 = -0.25,\ x_3 = 0.50,\ x_4 = 1.25,\ x_5 = 2 \quad T_4 = \sum_{i=0}^{4} \frac{\Delta x}{2}\left[f(x_i) + f(x_{i+1})\right]$

$T_4 = \frac{0.75}{2}\left[f(-1) + 2f(-.25) + 2f(.5) + f(1.25) + f(2)\right]$

$= \frac{.75}{2}\left[-16 + 2(-2.6) + 2(8) + 2(14) + (-26)\right] = \boxed{-1.2 \text{ un}^2}$

✓

(c) (5 pts) Calculate the Simpson's rule estimate S_4 for $\int_{-1}^{2} f(x)\,dx$.

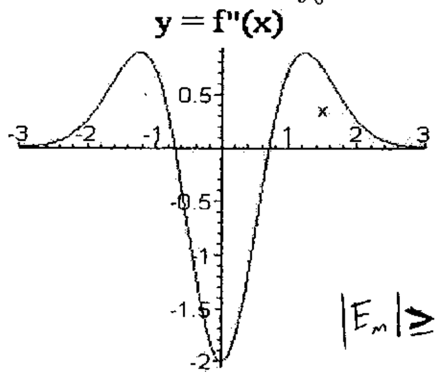
$n=4 \quad \Delta x = \frac{2-(-1)}{4} = 0.75 \quad S_n = \sum_{i=0}^{n} \frac{\Delta x}{3}\left[f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right]$

$\frac{0.75}{3}\left[f(-1) + 4f(-.5) + 2f(0) + 4f(.5) + 2f(1) + 4f(1.5) + f(2)\right]$

$.25\left[-16 + 4(-6) + 2(1) + 4(8) + 2(14) + 4(11) + (26)\right] = \boxed{10} \quad = 4.9$

-1

6) (10 pts) Let $f(x) = e^{-x^2}$. Find the least value of n which guarantees that the midpoint approximation rule for $\int_0^2 f(x)\,dx$ is accurate to within .01.



$f'(x) = e^{-x^2}(-2x)$

$f''(x) = e^{-x^2}(-2) + (-2x)e^{-x^2}(-2x)$

$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$

$|f''(x)| \le 2$ on $[0,$

$\therefore\ K = 2$

$|E_m| \geq \frac{2(2-0)^3}{24n^2}$ $\quad .01 \geq \frac{2(2)^3}{24n^2}$ $\quad n^2 \geq \frac{2^4}{24(.1)}$

-2

$n \geq 2.58$

$\boxed{n = 3}$ $\quad n \geq 9$

7) (10 pts) This problem concerns the curve defined by the parametric equations $x = 3t^2 - 9$, $y = t^3 - 3t$.

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(a) (5 pts) Determine the coordinates of all points on the curve where the tangent line to the curve is horizontal.

Horizontal when $\frac{dy}{dt} = 0$ $\quad \frac{dx}{dt} \neq 0$ $\quad \frac{dx}{dt} = 6t$ $\quad \frac{dy}{dt} = 3t^2 - 3$

or $\frac{dy}{dx} = 0$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\quad \frac{dy}{dx} = \frac{3t^2-3}{6t} = \frac{t^2-1}{2t} = 0$ $\quad t^2 - 1 = 0$, $t^2 = 1$, $t = \pm 1$ $\quad \frac{dx}{dt} \neq 0$ @ $t = \pm 1$ $\quad \therefore$ $\boxed{t = \pm 1}$

(b) (5 pts) Determine all values of t where the curve is concave upward and determine all values of t where the curve is concave downward.

$\frac{dy}{dx} = \frac{t^2-1}{2t}$ $\quad \frac{dy^2}{d^2x} = \frac{\frac{(2t)(2t) - (t^2-1)(2)}{(2t)^2}}{\frac{6t}{1}} = \frac{4t^2 - 2t^2 + 2}{24t^3} = \frac{2t^2+2}{24t^3} = \boxed{\frac{t^2+1}{12t^3}}$

Concave up when $\frac{t^2+1}{12t^3} > 0$ $\quad t^2+1 > 0\ \forall t$ $\quad \therefore$ Concave up when $12t^3 > 0$

concave down when $\frac{t^2+1}{12t^3} < 0$ $\quad t^2+1 > 0\ \forall t$ $\quad \therefore$ concave down when $12t^3 < 0$

$12t^3 > 0$, $t^3 > 0$, $t > 0$ $\qquad 12t^3 < 0$, $t^3 < 0$, $t < 0$

Concave up $\forall\, t > 0$
Concave down $\forall\, t < 0$

8) (10 pts) It can be verified (but don't do it) that

10

$$\frac{2(x^2 - x + 1)}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{x-1}{x^2+1}$$

Use this to calculate $\int \frac{2(x^2 - x + 1)}{(x-1)(x^2+1)}\,dx.$ $= \int \frac{1}{x-1}\,dx + \int \frac{x-1}{x^2+1}\,dx = \int \frac{dx}{x-1} + \int \frac{x\,dx}{x^2+1} \int \frac{1}{x^2+1}\,dx$

$\int \frac{1}{x-1}\,dx = \ln|x-1|$ $\quad \int \frac{x}{x^2+1}\,dx$ $\quad \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$ $\quad \int \frac{1}{x^2+1}\,dx = \arctan(x)$

$d[\ln|x-1|] = \frac{1}{x-1}(1)\,dx$ $\quad u = x^2+1$ $\quad = \frac{1}{2}\ln|x^2+1|$

$du = 2x\,dx$

$$\int \frac{dx}{x-1} + \int \frac{x\,dx}{x^2+1} - \int \frac{dx}{x^2+1} = \boxed{\ln|x-1| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C}$$

9) (8 pts) Calculate the partial fraction expansion for the function $f(x) = \dfrac{3x^2 - 3x + 1}{x^4 + x^2}$. Determine the values of any constants you introduce.

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$$f(x) = \left[\frac{3x^2-3x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}\right] x^2(x^2+1)$$

$$3x^2-3x+1 = A(x)(x^2+1) + B(x^2+1) + (Cx+D)(x^2)$$

$$3x^2-3x+1 = A(x^3+x) + B(x^2+1) + Cx+D(x^2)$$

$$0x^3 + 3x^2 - 3x + 1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

$A+C=0$ $\quad B+D=3$ $\quad A=-3$ $\quad B=1$

$A=-C$ $\quad D=3-B$ $\quad A=-3$ $\quad B=1$

$-3=-C$ $\quad D=3-1$

$C=3$ $\quad D=2$ $\quad A=-3$ $\quad B=1$

$$\boxed{f(x) = \frac{-3}{x} + \frac{1}{x^2} + \frac{3x+2}{x^2+1}}$$

10) (9 pts) Calculate the integral $\int_2^\infty \frac{1}{x(x-1)}\,dx$.

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$$\left[\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}\right]x(x-1) \qquad 1 = A(x-1) + B(x)$$

$$-\int \frac{dx}{x} = -\ln(x)$$

$$-\ln(b) + \ln(\cdot)$$

$$\ln(b-1) - \ln(2)$$

$$0x + 1 = (A+B)x + (-A)$$

$$A+B=0 \qquad -A=1$$

$$-1+B=0 \qquad A=-1$$

$$B=1$$

$$\int_2^\infty -\frac{1}{x} + \frac{1}{x-1}\,dx$$

$$= \lim_{b\to\infty} \int_2^b -\frac{1}{x} + \frac{1}{x-1}\,dx = \lim_{b\to\infty}\left[-\ln(x) + \ln(x-1)\Big|_2^b\right]$$

$$= \lim_{b\to\infty}\left[(-\ln(b)+\ln(b-1)) - (-\ln(2)+\ln(2-1))\right]$$

$$= \lim_{b\to\infty}\left[\ln(b-1) - \ln(b) + \ln(2) - \ln(1)\right]$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{b-1}{b}\right) + \ln(2) - 0\right] = \ln(1) + \ln(2) = \boxed{\ln(2}$$